MATHEMATICS
Class : XII
TOPPER Sample Paper 1

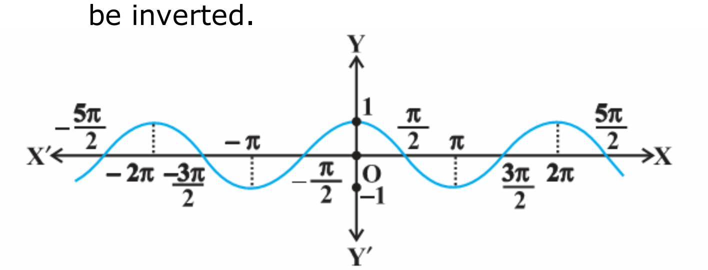
Time Allowed : 3 Hrs Maximum Marks: 100

1. All questions are compulsory.
2. The question paper consist of 29 questions divided into three sections A, B and C. Section A comprises of 10 questions of one mark each, section B comprises of 12 questions of four marks each and section C comprises of 07 questions of six marks each.
3. All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
4. There is no overall choice. However, internal choice has been provided in 04 questions of four marks each and 02 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
5. Use of calculators is not permitted.

SECTION – A

1. If $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ and a function $f : A \to B$ is given by $f = \{(a, 2), (b, 3), (c, 1)\}$. Is f a bijective function?

2. From the graph of y = cos x, identify the intervals of x in which the function can be inverted.



$y = \cos x$

3. If A and B are square matrices of the same order. Check whether $(A + B)^2 = A^2 + 2AB + B^2$ is true or not

4. If a matrix $A = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}$, Show that $(A - A')$ is a skew symmetric matrix, where A' is the transpose of matrix A.

5. If $A = \begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}$ and $A^2 = \begin{pmatrix} -5 & -18 \\ 18 & 7 \end{pmatrix}$ find $A^2 - 6A + 17I$.

6. Find $\int \frac{1}{\sqrt{9 - 25x^2}} dx$

7. Find a unit vector in the direction of $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$.

8. Find the components and magnitude of the vector $\overrightarrow{PQ}$, where P has coordinates (-1,-2,4) and Q has coordinates (2,0,-2).
9. Show that the points A(3,-5,1) , B (-1, 0 , 8) and C (7 , -10 , -6) are collinear.
10. Find $\int_{-\pi/2}^{\pi/2} \sin^7 x dx$

SECTION – B

11. Let A = Q x Q , Q being the set of rationals . Let '*' be a binary operation on A , defined by (a, b) * (c , d) = (ac , ad + b) . Show that
(i) '*' is not commutative (ii) '*' is associative
(iii The Identity element w.r.t '*' is (1 , 0)

OR

Let '*'be a binary operation on the set { 0,1,2,3,4,5} and

$$a*b = \begin{cases} a+b \text{ if } & a+b<6 \\ a+b-6 \text{ if } & a+b \ge 6 \end{cases}$$

Find the identity element and the inverse element of each element of the set for the operation '*'.

12. Solve the Equation

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1} x, (x > 0)$$

13. Is the given function continuous at x = 0?

$$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

14.Find $\frac{dy}{dx}$, when $y = \sqrt{a+\sqrt{a+\sqrt{a+x^2}}}$, where a is a constant .

15. Discuss the applicability of Lagrange's Mean value theorem for the function:

$f(x) = |\sin x|$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

16. Evaluate the integral $\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$

OR

Evaluate $\int_0^{\pi/2} \sin 2x \ \tan^{-1}(\sin x) dx$

17.Form the differential equation satisfied by $\sqrt{(1-x^2)}+\sqrt{(1-y^2)} = =$ a.(x - y) where a is an arbitrary constant.

18. Solve the differential equation (x-1)dy + y dx = x (x-1) y $^{1/3}$ dx

OR

Solve the differential equation: $\sec^2 y\frac{dy}{dx}+2x\tan y = x^3$

19. Find the angle between $\vec{a}$ and $\vec{b}$. If $\vec{a}+\vec{b}+\vec{c}=0$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$ & $|\vec{c}| = 7$

OR

Find λ if the vectors $\vec{a} = \hat{i}-\lambda\hat{j}+3\hat{k}$ and $\vec{b} = 4\hat{i}-5\hat{j}+2\hat{k}$ are perpendicular to each other.

20. Find the distance of the point (2, 4, -1) from the line

$$\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$$

21. A die is tossed thrice. Find the probability of getting an odd number at least once.

22. Find the interval in which the value of the determinant of the matrix A lies.

Given A= $\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$

SECTION – C

23. Find A^{-1}, by using elementary row transformations . Given $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$.

24. Show that the height of the cylinder of greatest volume that can be inscribed in a right circular cone of height h and having semi vertical angle α is one third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3\tan^2\alpha$

25. Prove that the curves $y = x^2$ and $x = y^2$ divide the square bounded by x = 0 , y=0 , x=1 and y=1 into three parts that are equal in area.

26. The chances of a patient having a heart attack is 40%. According to latest research Drug A reduces the risk of heart attack by 30% and drug B reduces

its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random
Suffers a heart attack. Find the probability that the patient had been prescribed Drug A.

OR

A factory manufactures screws, machines X, Y and Z manufacture respectively 1000,2000, 3000 of the screws,1%, 1.5% and 2 % of their outputs are respectively defective .A screw is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine X ?

27. Find the equation of two lines through the origin which intersect the line

$$\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1} \text{ at angles of } \frac{\pi}{3}.$$

28. Find $\int \frac{x^4.dx}{\left(x-1)(x^2+1)\right)}$

OR

Find $\int(\sqrt{\cot x}+\sqrt{\tan x}).dx$

29. Two tailors A and B earn Rs 150 and Rs 200 per day respectively. A can stitch 6 shirts and 4 pants per day while B can stitch 10 shirts and 4 pants per day . How many days shall each work , if it is desired to produce atleast 60 shirts and 32 pants at a minimum labour cost ? Solve the problem graphically.

Class XII
Marking Scheme
Model Paper I

Section A

1. If $A = \{ a, b, c \}$ and $B = \{ 1, 2, 3 \}$ and a function $f : A \rightarrow B$ is given by $f = \{ (a, 2), (b, 3), (c, 1) \}$

Every element of set A is mapped to the unique element of set B.i.e each element in the set B has a unique pre image in B

$\Rightarrow$ f is a one – one function

Range of $f = \{1, 2, 3\} = B$
$\Rightarrow$ f is an onto function
$\therefore$ f is a injective function
[1 Mark]

2. The function $y = \cos x$ can be inverted in the intervals where it is both one –one and onto i.e in the intervals
$[-2\pi, -\pi], [-\pi, 0], [0, \pi], [\pi, 2\pi]$
[1 Mark]

3. $(A + B)^2 = (A + B)(A + B) = A(A + B) + B(A + B)$
$= A^2 + AB + BA + B^2$ which may or may not be equal to $A^2 + 2AB + B^2$
[Since matrix multiplication is not commutative]

So the expression is not true in general. [1 Mark]

4. Let $A = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$

$A - A' = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$

Transpose of $(A - A') = (A - A')'$

$= \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix} = -(A - A')$

$\Rightarrow (A - A')$ is a skew symmetric matrix (1 Mark)

5. $A = \begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}$ and $A^2 = \begin{pmatrix} -5 & -18 \\ 18 & 7 \end{pmatrix}$

$$A^2 - 6A = \begin{pmatrix} -5 & -18 \\ 18 & 7 \end{pmatrix} - 6\begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix}$$

$$A^2 - 6A = \begin{pmatrix} -5 & -18 \\ 18 & 7 \end{pmatrix} - \begin{pmatrix} 12 & -18 \\ 18 & 24 \end{pmatrix} = \begin{pmatrix} -17 & 0 \\ 0 & -17 \end{pmatrix}$$

$$A^2 - 6A + 17I = \begin{pmatrix} -17 & 0 \\ 0 & -17 \end{pmatrix} + 17\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -17 & 0 \\ 0 & -17 \end{pmatrix} + \begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix}$$

$$A^2 - 6A + 17\,I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

[1 Mark]

6. $I = \int \frac{1}{\sqrt{9 - 25x^2}} dx$

$= \frac{1}{5} \int \frac{1}{\sqrt{\left(\frac{3}{5}\right)^2 - x^2}} dx$

$= \frac{1}{5} \sin^{-1} \left(\frac{x}{\frac{3}{5}} \right) + c$

$= \frac{1}{5} \sin^{-1} \left(\frac{5x}{3} \right) + c$

[1 Mark]

7. The unit vector in the direction of vector $\vec{a} = \hat{a} = \frac{\vec{a}}{|\vec{a}|}$

$\therefore \ \hat{a} = \frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{|3\hat{i} - 2\hat{j} + 6\hat{k}|} = \frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{\sqrt{9 + 4 + 36}}$

$= \frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{\sqrt{49}} = \frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{7} = \frac{3}{7}\hat{i} - \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}$

[1 Mark]

8. P(-1,-2,4) and Q(2,0,-2)

position vector of P = $-1\hat{i}-2\hat{j}+4\hat{k}$

position vector of Q = $2\hat{i}+0\hat{j}-2\hat{k}$

$\overrightarrow{PQ} =$ position vector of Q - position vector of P $= \left(2\hat{i}+0\hat{j}-2\hat{k}\right)-\left(-1\hat{i}-2\hat{j}+4\hat{k}\right)$

$= 3\hat{i}+2\hat{j}-6\hat{k}$ $(\frac{1}{2}$ mark)

Magnitude of $\overrightarrow{PQ} = \sqrt{3^2+2^2+(-6)^2} = \sqrt{49} = 7$ $(\frac{1}{2}$ mark)

9. Given A(3,-5,1), B (-1, 0 , 8) and C (7 , -10 , -6)

$\therefore$ position vector of A = $3\hat{i}-5\hat{j}+1\hat{k}$

position vector of B = $-1\hat{i}+0\hat{j}+8\hat{k}$

position vector of C = $7\hat{i}-10\hat{j}-6\hat{k}$

$\overrightarrow{AB} =$ position vector of B - position vector of A $= \left(-1\hat{i}+0\hat{j}+8\hat{k}\right)-\left(3\hat{i}-5\hat{j}+1\hat{k}\right)$

$= -4\hat{i}+5\hat{j}+7\hat{k}$

$\overrightarrow{AC} =$ position vector of C -position vector of A $= \left(7\hat{i}-10\hat{j}-6\hat{k}\right)-\left(3\hat{i}-5\hat{j}+1\hat{k}\right)$

$= 4\hat{i}-5\hat{j}-7\hat{k}$

$\overrightarrow{AC} = -\overrightarrow{AB}$

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ has same magnitude but opposite directions

$\Rightarrow$ The points A, B and C are collinear [1 Mark]

10. Let $f(x) = \sin^7 x$

$f(-x) = \sin^7(-x) = -\sin^7 x = -f(x)$

So f(x) is an odd function of x

$\therefore \int_{-\pi/2}^{\pi/2} \sin^7 x.dx = 0$ [1 Mark]

Section B : Section B comprises of 12 questions of four marks each .

11.(i) $(a, b) * (c, d) = (ac, ad + b)$
$(c, d) * (a, b) = (ca, cb + d)$
$(ac, ad + b) \neq (ca, cb + d)$
So, '*' is not commutative

[1 Mark]

(ii) Let (a,b) (c, d), $(e, f) \in A$, Then
$((a, b)*(c, d)) * (e,f) = (ac, ad + b) * (e,f) = ((ac)e, (ac)f + (ad+b))$
$= (ace, acf + ad + b)$
$(a,b)*((c,d)*(e,f)) = (a,b) * (ce, cf + d) = (a(ce), a(cf+d) + b) = (ace, acf + ad + b)$

$((a, b)*(c, d)) * (e,f) = (a,b)((c,d)*(e,f))$
Hence, '*' is associative. [1 Mark]

(iii) Let $(x, y) \in A$. Then (x, y) is an identity element, if and only if
$(x, y) * (a, b) = (a, b) = (a,b) * (x, y)$, for every $(a,b) \in A$
Consider $(x, y) * (a, b) = (xa, xb + y)$
$(a,b) * (x, y) = (ax, ay + b)$
$(xa, xb + y) = (a, b) = (ax, ay + b)$
[1 Mark]
$ax = xa = a \Rightarrow x = 1$
$xb + y = b = ay + b \Rightarrow b + y = b = ay + b \Rightarrow y = 0 = ay \Rightarrow y = 0$
Therefore , (1, 0) is the identity element
[1 Mark]

OR

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

[2 Marks]
From the table, the second row and second column are the same as the original set.
0*0 = 0 , 1*0 = 0*1 = 1 , 2*0= 0*2 = 2 , 3*0= 0*3 = 3 , 4*0 = 0*4 =4, 0*5 = 5*0 = 5
'0'is the identity element of the operation '*'
[1 Mark]

Now, the element '0' appears in the cell 1*5 = 5*1 =0, 2*4 = 4 *2= 0, 3*3 =0, and 0* 0 = 0
Inverse element of 0 is 0, Inverse element of 1 is 5, Inverse element of 2 is 4, Inverse element of 3 is 3, Inverse element of 4 is 2, Inverse element of 5 is 1.

[1 Mark]

12.Given: $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x \ (x>0)$

$$\Rightarrow 2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1}x$$

$$\Rightarrow \tan\left[2\tan^{-1}\left(\frac{1-x}{1+x}\right)\right] = \tan\left[\tan^{-1}x\right]$$ [1 Mark]

$$\Rightarrow \frac{2\tan\left[\tan^{-1}\left(\frac{1-x}{1+x}\right)\right]}{1-\left(\tan\left[\tan^{-1}\left(\frac{1-x}{1+x}\right)\right]\right)^2} = x$$ [1 Mark]

$$\Rightarrow \frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2} = x$$

$$\Rightarrow \frac{2(1-x)(1+x)}{(1+x)^2-(1-x)^2} = x$$ [1 Mark]

$$\Rightarrow \frac{(1-x^2)}{2x} = x$$

$$\Rightarrow 3x^2 = 1$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$$ [1 Mark]

13. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = \frac{0-1}{0+1} = -1$ [1 Mark]

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = \lim_{x\to 0^+} \frac{1 - \frac{1}{e^{\frac{1}{x}}}}{1 + \frac{1}{e^{\frac{1}{x}}}} = \frac{1-0}{1+0} = 1$ [1 Mark]

$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

So $\lim_{x\to 0} f(x)$ does not exist . [1 Mark]

$\Rightarrow$ f(x) is not continuous at x = 0 [1 Mark]

14. $y = \sqrt{a+\sqrt{a+\sqrt{a+x^2}}}$, where a is a constant .

$\Rightarrow y = \left[a + \sqrt{a+\sqrt{a+x^2}}\right]^{\frac{1}{2}}$ [1Mark]

$\frac{dy}{dx} = \frac{1}{2}\left[a + \sqrt{a+\sqrt{a+x^2}}\right]^{-\frac{1}{2}} \frac{d}{dx}\left[a + \sqrt{a+\sqrt{a+x^2}}\right]$ [1Mark]

$\frac{dy}{dx} = \frac{1}{2}\left[a + \sqrt{a+\sqrt{a+x^2}}\right]^{-\frac{1}{2}} \left[\frac{1}{2}\left(a + \sqrt{a+x^2}\right)^{-\frac{1}{2}}\right] \frac{d}{dx}\left(a + \sqrt{a+x^2}\right)$ [1Mark]

$\frac{dy}{dx} = \frac{1}{2}\left[a + \sqrt{a+\sqrt{a+x^2}}\right]^{-\frac{1}{2}} \left[\frac{1}{2}\left(a + \sqrt{a+x^2}\right)^{-\frac{1}{2}} \frac{1}{2}\left(a+x^2\right)^{-\frac{1}{2}} .2x\right]$ [1Mark]

$\frac{dy}{dx} = \frac{1}{4} x \left[\left(a + \sqrt{a+\sqrt{a+x^2}}\right).\left(a + \sqrt{a+x^2}\right).\left(a+x^2\right)\right]^{-\frac{1}{2}}$

15. $f(x) = |\sin x|$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Let $h(x) = \sin x, g(x) = |x|$

$\therefore goh(x) = f(x) = |\sin x|$

$h(x) = \sin x$ is a continuous function in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$g(x) = |x|$ is a continuous function in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore goh(x) = |\sin x|$ is also a continuous function in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ [1 Mark]

$h(x) = \sin x$ is a differentiable function in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$g(x) = |x|$ is not a differentiable function in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore goh(x) = |\sin x|$ is also not a differentiable function in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ [1 Mark]

$\therefore$ Conditions of Lagrange's theorem are not satisfied [1 Mark]

$\therefore$ Lagrange's theorem is not applicable for the given function [1 Mark]

16. $$\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}} = \int_0^1 \frac{\left(\sqrt{1+x} + \sqrt{x}\right)dx}{\left(\sqrt{1+x} - \sqrt{x}\right)\left(\sqrt{1+x} + \sqrt{x}\right)}$$

$$= \int_0^1 \frac{\left(\sqrt{1+x} + \sqrt{x}\right)dx}{1 + x - x}$$

$$= \int_0^1 \left(\sqrt{1+x} + \sqrt{x}\right)dx$$ [1 Mark]

$$= \left[\frac{(1+x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 + \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$$ [2 Marks]

$$= \frac{2}{3}\left[2^{\frac{3}{2}} - 1 + 1\right]$$

$$= \frac{2}{3}\left[2^{\frac{3}{2}}\right] = \frac{4\sqrt{2}}{3}$$ [1 Mark]

OR

$$\text{Let} I = \int_0^{\pi/2} \sin 2x \ \tan^{-1}(\sin x)dx$$

$$= \int_0^{\pi/2} 2 \sin x \cos x \tan^{-1}(\sin x)dx$$

$\text{Let} \sin x = t \Rightarrow \cos x dx = dt$

$x = 0 \Rightarrow t = 0, x = \frac{\pi}{2} \Rightarrow t = 1$

$\therefore I = \int_0^1 2t \tan^{-1} t \, dt = 2\int_0^1 t \tan^{-1} t \, dt$ [1 Mark]

Integrating by parts , we have

$$=2\left[\frac{t^2}{2}\tan^{-1} t\right]_0^1 - 2\int_0^1 \frac{t^2}{2(1+t^2)}dt$$

$$= 2\left[\frac{1}{2}\tan^{-1} 1 - 0\right] - \int_0^1 \frac{t^2}{(1+t^2)}dt$$

$= 2\left[\frac{1}{2} \times \frac{\pi}{4}\right] - \int_0^1 \frac{t^2+1-1}{(1+t^2)}dt$ [1 Mark]

$$= \frac{\pi}{4} - \int_0^1 \left(1 - \frac{1}{(1+t^2)}\right)dt$$

$= \frac{\pi}{4} - [t]_0^1 + [\tan^{-1} t]_0^1$ [1 Mark]

$$= \frac{\pi}{4} - 1 + \frac{\pi}{4}$$

$= \frac{\pi}{2} - 1$ [1 Mark]

17.Let $x = \sin\alpha$ and $y = \sin\beta$, such that $\alpha, \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\therefore \sqrt{1-\sin^2\alpha} + \sqrt{1-\sin^2\beta} = a(\sin\alpha - \sin\beta)$$

$$\therefore \cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$$

$$2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2a\left(\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right)$$

$$\cos\frac{\alpha-\beta}{2} = a\left(\sin\frac{\alpha-\beta}{2}\right)$$ [$1\frac{1}{2}$ Marks]

$$\cot\frac{\alpha-\beta}{2} = a$$

$$\frac{\alpha-\beta}{2} = \cot^{-1} a$$

$$\alpha - \beta = 2\cot^{-1} a$$

$$\sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a$$ [$1\frac{1}{2}$ Marks]

Differentaiting w.r.t x , we have

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$ [1 Mark]

18. $(x-1)dy + y\,dx = x(x-1)\,y^{\frac{1}{3}}\,dx$

$$\Rightarrow (x-1)\frac{dy}{dx} + y = x(x-1)\,y^{\frac{1}{3}}$$

$$\Rightarrow \frac{1}{y^{\frac{1}{3}}}\frac{dy}{dx} + \frac{y^{\frac{2}{3}}}{x-1} = x$$

Let $y^{2/3} = t \Rightarrow \frac{2}{3}y^{-1/3}\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow \frac{3}{2}\frac{dt}{dx} + \frac{t}{x-1} = x$$

$$\Rightarrow \frac{dt}{dx} + \frac{2}{3}\frac{t}{x-1} = \frac{2}{3}x$$ [1 Mark]

This is a linear differential equation, whose integrating factor is

$$\text{IF} = e^{\int\frac{2}{3}\left(\frac{1}{x-1}\right)dx} = e^{\frac{2}{3}\log(x-1)} = e^{\log(x-1)^{2/3}} = (x-1)^{\frac{2}{3}}$$ [$\frac{1}{2}$ Mark]

$\therefore$ Solution of Differential equation is

$$t(x-1)^{2/3} = \int \frac{2}{3} x (x-1)^{2/3} dx + c \qquad [1 \text{ Mark}]$$

$$\Rightarrow y^{2/3} \ (x-1)^{2/3} = \int \frac{2}{3} x (x-1)^{2/3} dx + c$$

$$\Rightarrow y^{2/3} \ (x-1)^{2/3} = \frac{2}{3}\left[\frac{x(x-1)^{5/3}}{\frac{5}{3}} - \int \frac{(x-1)^{5/3} dx}{\frac{5}{3}}\right] + C$$

$$= \frac{2}{5} x (x-1)^{5/3} - \frac{2}{5} \frac{(x-1)^{8/3}}{\frac{8}{3}} + C \qquad [1 \text{ Mark}]$$

$$\Rightarrow y^{2/3} \ (x-1)^{2/3} = \frac{2}{5} x (x-1)^{5/3} - \frac{3}{20}(x-1)^{8/3} + C$$

$$\Rightarrow y^{2/3} = \frac{2}{5} x (x-1) - \frac{3}{20}(x-1)^2 + C (x-1)^{-2/3} \qquad [\frac{1}{2} \text{ Mark}]$$

OR

18. $\sec^2 y\dfrac{dy}{dx}+2x\tan y=x^3$

Let $\tan y=t \Rightarrow \sec^2 y\ \ dy\ =\ dt$

$\Rightarrow \dfrac{dt}{dx}+2tx=x^3$ [1 Mark]

This is a linear differential equation with integrating factor :

IF= $e^{\int 2x dx}=e^{x^2}$ [$\frac{1}{2}$ Mark]

Solution of the differential equation is given by

$t\ e^{x^2}=\int x^3 e^{x^2}dx+C$ [$\frac{1}{2}$ Mark]

To solve $\int x^3 e^{x^2}dx$

Let $x^2=z \Rightarrow 2x\ \ dx\ =\ dz$

$\Rightarrow \int x^3 e^{x^2}dx=\dfrac{1}{2}\int ze^z dz$

$=\dfrac{1}{2}\left[ze^z-\int e^z dz\right]+C$

$=\dfrac{1}{2}\left[ze^z-e^z\right]+C$

$=\dfrac{1}{2}\left(x^2-1\right)e^{x^2}+C$ [1 Mark]

$\Rightarrow t=\dfrac{1}{2}\left(x^2-1\right)+Ce^{-x^2}$

$\Rightarrow \tan y=\dfrac{1}{2}\left(x^2-1\right)+Ce^{-x^2}$ [1 Mark]

19. Here, $\vec{a}+\vec{b}+\vec{c}=0$

$\Rightarrow\ \vec{a}+\vec{b}=-\vec{c}$

$\Rightarrow \left(\vec{a}+\vec{b}\right)^2=\left(-\vec{c}\right)^2$ [$\frac{1}{2}$mark]

$\Rightarrow \vec{a}^2+\vec{b}^2+2\vec{a}.\vec{b}=\vec{c}^2$ [1 mark]

$\Rightarrow \left|\vec{a}\right|^2+\left|\vec{b}\right|^2+2\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta=\left|\vec{c}\right|^2$, [1 mark]

where θ is the angle between $\vec{a}$ and $\vec{b}$

$\Rightarrow (3)^2 + (5)^2 + 2(3)(5)\cos\theta = (7)^2$ (1 Mark)

$\Rightarrow 9 + 25 + 30\cos\theta = 49$

$\Rightarrow \cos\theta = \frac{15}{30} = \frac{1}{2}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^{o}$ [$\frac{1}{2}$ Mark]

OR

$\vec{a} = \hat{i} - \lambda\hat{j} + 3\hat{k}$ and $\vec{b} = 4\hat{i} - 5\hat{j} + 2\hat{k}$

vectors are perpendicular if $\vec{a}.\vec{b} = 0$ [1 mark]

$\vec{a}.\vec{b} = \left(\hat{i} - \lambda\hat{j} + 3\hat{k}\right).\left(4\hat{i} - 5\hat{j} + 2\hat{k}\right)$ [1 mark]

$= 1 \times 4 + (-\lambda) \times (-5) + 3 \times 2 = 4 + 5\lambda + 6 = 10 + 5\lambda$ [1 mark]

$\Rightarrow 10 + 5\lambda = 0 \Rightarrow \lambda = -2$ [1 mark]

20. The given line is $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$

Given point is (2, 4, -1)

The distance of a point whose position vector is $\overline{a_2}$ from a line whose vector equation is $\vec{r} = \bar{a}$

$$d = \frac{\left|\vec{v} \times (\vec{a_2} - \vec{a_1})\right|}{\left|\vec{v}\right|} \qquad (\tfrac{1}{2}\text{Mark})$$

$$= \frac{\left|\left(\hat{i} + 4\hat{j} - 9\hat{k}\right) \times \left((2\hat{i} + 4\hat{j} - 1\hat{k}) - (-5\hat{i} - 3\hat{j} + 6\hat{k})\right)\right|}{\left|\left(\hat{i} + 4\hat{j} - 9\hat{k}\right)\right|} \qquad (\tfrac{1}{2}\text{Mark})$$

$$= \frac{\left|\left(\hat{i} + 4\hat{j} - 9\hat{k}\right) \times (7\hat{i} + 7\hat{j} - 7\hat{k})\right|}{\left|\left(\hat{i} + 4\hat{j} - 9\hat{k}\right)\right|} \qquad (1\text{Mark})$$

$$\left(\hat{i} + 4\hat{j} - 9\hat{k}\right) \times (7\hat{i} + 7\hat{j} - 7\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & -9 \\ 7 & 7 & -7 \end{vmatrix} = 35\hat{i} - 56\hat{j} - 21\hat{k} \qquad (1\text{Mark})$$

$$= \frac{7}{\sqrt{98}}\left|\left(5\hat{i} - 8\hat{j} - 3\hat{k}\right)\right| = \frac{7}{\sqrt{98}}\sqrt{98} = 7 \text{ units} \qquad (1\text{Mark})$$

21. $S = \{(x,y,z) : x,y,z \in \{1,2,3,4,5,6\}\}$

S contains $6\times6\times6 = 216$ cases [1/2 mark]

Let E : an odd number appears atleast once E' : an odd number appears none of the times

i.e E' : an even number appears all three times [1 mark]

$E' = \{(x,y,z) : x,y,z \in \{2,4,6\}\}$

E' contains $3\times3\times3 = 27$ cases [1 mark]

Now, $P(E) = 1 - P(E')$ [1/2 mark]

$= 1 - \frac{27}{216} = 1 - \frac{1}{8} = \frac{7}{8}$ [1 mark]

22. $|A| = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

$= 1(1+\sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + 1(1+\sin^2\theta)$

$= 2(1+\sin^2\theta)$ [1 mark]

$0 \le \sin^2\theta \le 1$ [$\frac{1}{2}$mark]

$1 \le (1+\sin^2\theta) \le 2$

$2(1) \le 2(1+\sin^2\theta) \le 2(2)$

$2 \le 2(1+\sin^2\theta) \le 4$ [$1\frac{1}{2}$Marks]

$2 \le |A| \le 4$

$|A| \in [2,4]$ [1 mark]

So value of |A| is in interval [2,4]

Section C : Section C comprises of 07 questions of six marks each.

23

Given A= $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

A=IA

$$\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_2 - R_3$

$$\begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ (1 Mark)

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & -5 \\ 0 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ -3 & -3 & 4 \end{bmatrix} A$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & -5 & -10 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & -3 & 4 \\ -2 & -1 & 2 \end{bmatrix} A$$ [1 Mark]

$R_2 \to \frac{1}{-5}R_2, R_3 \to \frac{1}{-5}R_3$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ \frac{3}{5} & \frac{3}{5} & \frac{-4}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{-2}{5} \end{bmatrix} A$$ [1mark]

$R_2 \to R_2 - 2R_3, R_1 \to R_1 - 4R_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-3}{5} & \frac{1}{5} & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & \frac{-2}{5} \end{bmatrix} A$$ [1mark]

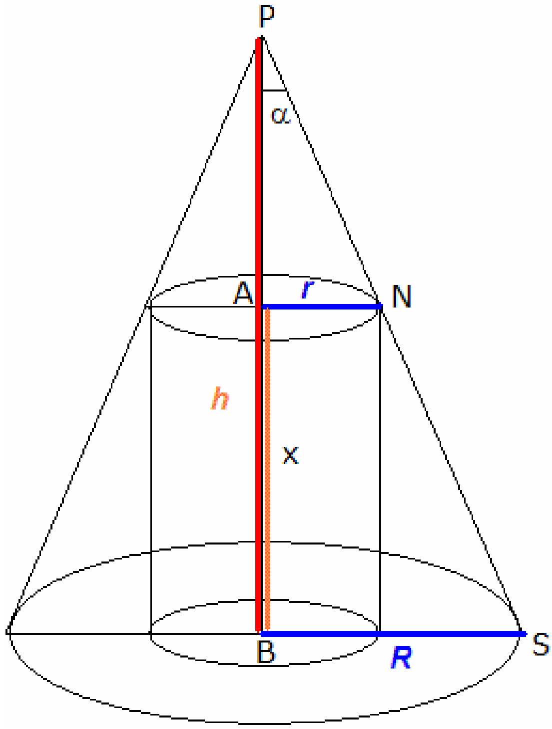
$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ \frac{-1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & \frac{-2}{5} \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix} A$$ (1 mark)

$$A^{-1} = -\frac{1}{5}\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$$ [1 mark]

24.



Let radius of the cone be R and height h. Let r be the radius of the cylinder and x be its height.

Consider triangles PAN and PBS. !PAN ~ !PBS

$$\frac{PA}{PB} = \frac{AN}{BS}$$

$$\Rightarrow \frac{h-x}{h} = \frac{r}{R}$$

$$r = \frac{R(h-x)}{h}$$

The volume of the cylinder

$$V = \pi r^2 x$$

$$= \pi \left[\frac{R(h-x)}{h}\right]^2 x$$

$$= \frac{\pi R^2 (h-x)^2 x}{h^2}$$

$$= \frac{\pi R^2 \left(h^2 x + x^3 - 2hx^2\right)}{h^2} \qquad \left(1\frac{1}{2} \text{ Marks}\right)$$

$$\frac{dV}{dx}=\frac{\pi R^2\left(h^2+3x^2-4hx\right)}{h^2}$$

$$\frac{dV}{dx}=0$$

$$\Rightarrow \frac{\pi R^2\left(h^2+3x^2-4hx\right)}{h^2}=0$$

$$\Rightarrow \left(3x^2+h^2-4hx\right)=0$$

$$\Rightarrow (3x-h)(x-h)=0$$

$$\Rightarrow x=h,\ \frac{h}{3},$$

but $x<h$, so $h=\frac{h}{3}$ [$1\frac{1}{2}$ marks]

$$\frac{d^2V}{dx^2}=\frac{\pi R^2(6x-4h)}{h^2}$$ [1 mark]

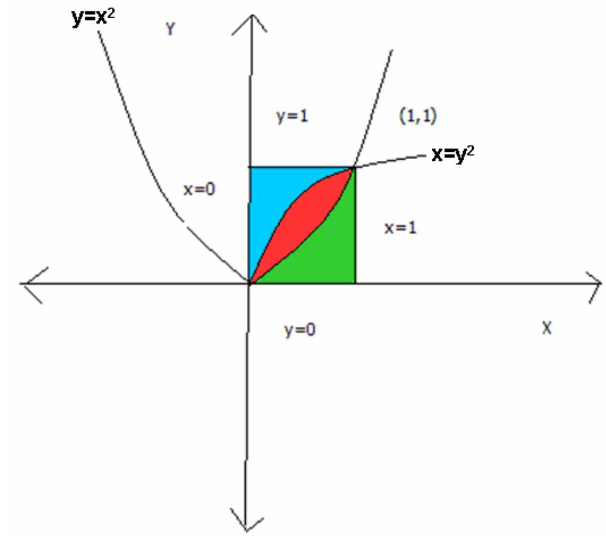
$$\left.\frac{d^2V}{dx^2}\right]_{x=\frac{h}{3}}=\left.\frac{\pi R^2(6x-4h)}{h^2}\right]_{x=\frac{h}{3}}=-\frac{2\pi R^2}{h}<0$$ [1 mark]

x=h/3 is a point of local maxima

$$V\Big]_{x=h/3}=\left.\frac{\pi R^2(h-x)^2x}{h^2}\right]_{x=\frac{h}{3}}$$

$$=\frac{4\pi R^2h}{27}=\frac{4\pi h^3\tan^2\alpha}{27}\left[\because \frac{R}{h}=\tan\alpha\right]$$ [1 mark]

25.



[1 mark]

The points where the two parabolas meet in the first quadrant are obtained by solving the two equations $y = x^2$ ....(1) and $x = y^2$. ....(2)

Substituting from (2) into (1) , we get

$X = (x^2)^2$

$\Rightarrow x = x^4$

$\Rightarrow \quad x^4 - x = 0$, i.e $x(x^3 - 1) = 0 \quad \Rightarrow \quad x = 0, 1$

So $y = 0,1$

$\therefore$The points where the two parabolas meet in the first quadrant are (0,0) and (0, 1).

The area gets divided into 3 parts as shown in three different colours.

[1 Mark]

Area I (In Blue)= $\int_0^1 \left(1-\sqrt{x}\right)dx = \left[x - \frac{x^{3/2}}{3/2}\right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$ squnits

Area II (In Red)= $\int_0^1 \left(\sqrt{x} - x^2\right)dx = \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1 = \left(\frac{2}{3} - \frac{1}{3}\right) = \frac{1}{3}$ squnits

Area III (In Green)= $\int_0^1 \left(x^2\right)dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$ sq.units

Area I =Area II =Area III

$\therefore$The curves $y = x^2$ and $x = y^2$ divide the square bounded by $x = 0$, $y=0$, $x=1$ and $y=1$ into three parts that are equal in area.

[Each area 1 mark , conclusion 1 mark]

26 Let E_1 be the event that a patient used Drug A$\therefore P(E_1) = ½$

Let E_2 be the event that a patient used Drug B$\therefore P(E_2) = ½$

Let E be the event that a patient had a heart attack

Required probability : $P(E_1/E)$
[1 mark]

$$P(E/E_1) = \frac{40}{100}\left(1 - \frac{30}{100}\right) = \frac{28}{100}$$

$$P(E/E_2) = \frac{40}{100}\left(1 - \frac{25}{100}\right) = \frac{30}{100}$$ [2 marks]

$$P(E_1/E) = \frac{P(E/E_1)P(E_1)}{P(E/E_1)P(E_1) + P(E/E_2)P(E_2)}$$ [1 mark]

$$= \frac{\frac{28}{100} \times \frac{1}{2}}{\frac{28}{100} \times \frac{1}{2} + \frac{30}{100} \times \frac{1}{2}} = \frac{14}{29}$$ [2 marks]

OR

S_1 : the bulb is manufactured by machine X
S_2 : the bulb is manufactured by machine Y
S_3 : the bulb is manufactured by machine Z
Required probability : $P(S_1/E)$
[1 mark]

$P(S_1) = 1/6$
$P(S_2) = 1/3$
$P(S_3) = ½$

$$P(E \mid S_1) = \frac{1}{100}$$
,
$$P(E \mid S_2) = \frac{3}{200}$$

$$P(E \mid S_3) = \frac{2}{100}$$ [2 marks]

$$P(S_1|E)=\frac{P(S_1)(P(E|S_1)}{P(S_1)(P(E|S_1))+P(S_2)(P(E|S_2))+P(S_3)(P(E|S_3))}$$ [1 mark]

$$=\frac{\frac{1}{6}\times\frac{1}{100}}{\frac{1}{6}\times\frac{1}{100}+\frac{1}{3}\times\frac{3}{200}+\frac{1}{2}\times\frac{2}{100}}$$

$$=\frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{2}+1}$$

$$=\frac{1}{1+3+6}=\frac{1}{10}$$ [2 marks

27. Given equation of line : $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$

Let P be any point on the line , then P(3+2t, 3+t , t)

Now OP makes an angle $\frac{\pi}{3}$, with the given line

$$\cos\frac{\pi}{3}=\frac{|2(3+2t)+1.(3+t)+1.t|}{\sqrt{6}\sqrt{(3+2t)^2+(3+t)^2+t^2}}$$ [1 mark]

$$\frac{1}{2}=\frac{|6t+9|}{\sqrt{6}\sqrt{6t^2+18t+18}}$$ [1 mark]

Squaring and simplifying , we get

$\Rightarrow t^2+3t+2=0$

$\Rightarrow (t+1)(t+2)=0$

$\Rightarrow t=-1, t=-2$ [1 mark]

$\Rightarrow$ The points through which the lines pass are $P(1,2,-1)$ and $P(-1,1,-2)$

$\Rightarrow$ The lines are

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{-1} \text{ and } \frac{x}{-1}=\frac{y}{1}=\frac{z}{-2}$$ [2 marks]

28.

$$\frac{x^4}{(x-1)(x^2+1)}=(x+1)+\frac{1}{x^3-x^2+x-1}$$

$$=(x+1)+\frac{1}{(x-1)(x^2+1)} \qquad \dots(1)$$

[1 mark]

$$\frac{1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)} \qquad \ldots (2)$$

[11/2 marks]

$$1 = A(x^2+1) + (Bx + C)(x-1)$$
$$= (A + B)x^2 + (C - B)x + A - C$$

$A + B = 0$, $C - B = 0$ and $A - C = 1$,

$$A = \frac{1}{2}, B = C = -\frac{1}{2}$$

[1 mark]

$$\frac{1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} - \frac{1}{2}\frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)} \qquad \ldots (3)$$

$$\frac{x^4}{(x-1)(x^2+x+1)} = (x+1) + \frac{1}{2(x-1)} - \frac{1}{2}\frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)}$$

[11/2 marks]

$$\int \frac{x^4}{(x-1)(x^2+x+1)}dx = \int (x+1)\,dx + \int \frac{1}{2(x-1)}dx - \int \frac{x}{2(x^2+1)}dx - \int \frac{1}{2(x^2+1)}dx$$

$$\int \frac{x^4}{(x-1)(x^2+x+1)}dx = \frac{x^2}{2} + x + \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + C$$

[1 mark]

OR

$$I = \int \left[\sqrt{\cot x} + \sqrt{\tan x}\right] dx = \int \sqrt{\tan x}(1 + \cot x)\, dx$$

Put $\tan x = t^2$, so that $\sec^2 x\, dx = 2t\, dt$

or $$dx = \frac{2t\, dt}{1 + t^4}$$

Then $$I = \int t\left(1 + \frac{1}{t^2}\right)\frac{2t}{(1 + t^4)}\, dt$$

[1 Mark]

$$= 2\int \frac{(t^2 + 1)}{t^4 + 1}\, dt = 2\int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t^2 + \frac{1}{t^2}\right)} = 2\int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 2}$$

[1 Mark]

Put $t - \frac{1}{t} = y$, so that $\left(1 + \frac{1}{t^2}\right) dt = dy$. Then

[1 Mark]

$$I = 2\int \frac{dy}{y^2 + \left(\sqrt{2}\right)^2} = \sqrt{2}\tan^{-1}\frac{y}{\sqrt{2}} + C = \sqrt{2}\tan^{-1}\frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}} + C$$

$$= \sqrt{2}\tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}\, t}\right) + C = \sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + C$$

[3 Marks]

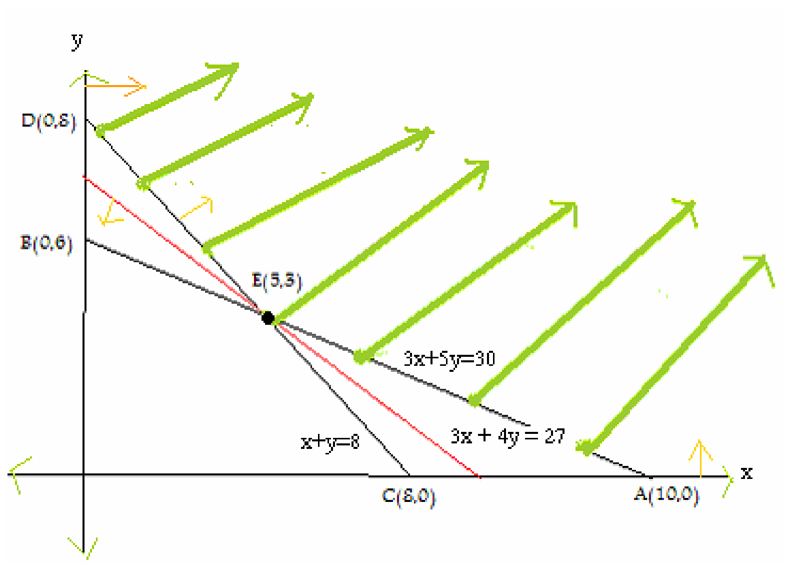
29.Let the two tailors work for x days

and y days respectively,
The problem is to minimise
the objective function
C= 150 x + 200 y
Subject to the constraints
$6x + 10y \geq 60 \Leftrightarrow 3x + 5y \geq 30$
$4x+4y \geq 32 \Leftrightarrow x + y \geq 8$
And

$x \geq 0, y \geq 0$

[2 Marks]

Feasible region is shown shaded .
.



[2 marks]
This region is unbounded

.

corner points	objective func
A(10,0)	1500
E(5,3)	1350.
D(0,8)	1600

[1 Mark]

The red line in the graph shows the line 150 x + 200y = 1350 or 3x + 4y =27

We see that the region 3x + 4y > 27 has no point in common with the feasible region.

Hence, the function has minimum value at E (5,3).

Hence, The labour cost is the least, when tailor A works for 5 days and Tailor B works f 3 days,

[1mark]