Class XII: Math

Chapter: Relations and Functions

Concepts and Formulae

Key Concepts

1. A relation R between two non empty sets A and B is a subset of their Cartesian Product $A \times B$. If A = B then relation R on A is a subset of $A \times A$

2. If (a, b) belongs to R, then a is related to b, and written as a R b If (a, b) does not belongs to R then a $\not R$ b.

3. Let R be a relation from A to B.
   Then Domain of $R \subset A$ and Range of $R \subset B$ co domain is either set B or any of its superset or subset containing range of R

4. A relation R in a set A is called **empty** relation, if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

5. A relation R in a set A is called **universal** relation, if each element of A is related to every element of A, i.e., $R = A \times A$.

6. A relation R in a set A is called

   a. **Reflexive**, if $(a, a) \in R$, for every $a \in A$,

   b. **Symmetric**, if $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

   c. **Transitive**, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, or all $a_1, a_2, a_3 \in A$.

7. A relation R in a set A is said to be an **equivalence relation** if R is reflexive, symmetric and transitive.

8. The empty relation R on a non-empty set *X* (i.e. a R b is never true) is not an equivalence relation, because although it is vacuously symmetric and transitive, it is not reflexive (except when *X* is also empty)

9. Given an arbitrary equivalence relation R in a set X, R divides X into mutually disjoint subsets S_i called partitions or subdivisions of X satisfying:

   - All elements of S_i are related to each other, for all i

- No element of S_i is related to S_j ,if $i \neq j$
- $\bigcup_{i=1}^{n} S_j$ =X and $S_i \cap S_j = \phi$, if $i \neq j$
- The subsets S_j are called Equivalence classes.

10. A function from a non empty set A to another non empty set B is a correspondence or a rule which associates every element of A to a unique element of B written as
$f: A \to B$ s.t f(x) = y for all $x \in A$, $y \in B$. All functions are relations but converse is not true.

11. If f: $A \to B$ is a function then set A is the domain, set B is co-domain and set $\{f(x): x \in A\}$ is the range of f. Range is a subset of codomain.

12. f: $A \to B$ is one-to-one if

    For all x, $y \in A$ $f(x) = f(y) \Rightarrow x = y$ or $x \neq y \Rightarrow f(x) \neq f(y)$

    A one- one function is known as injection or an Injective Function.

    Otherwise, f is called many-one.

13. f: $A \to B$ is an onto function ,if for each $b \in B$ there is atleast one $a \in A$ such that f(a) = b

    i.e if every element in B is the image of some element in A, f is onto.

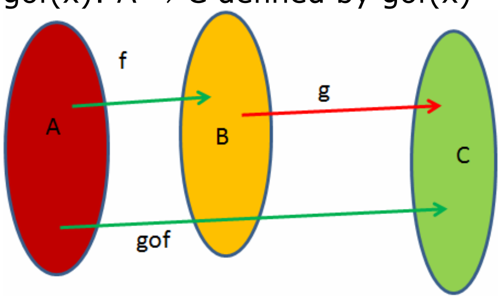
14. A function which is both one-one and onto is called a bijective function or a bijection.

15. For an onto function range = co-domain.

16. A one – one function defined from a finite set to itself is always onto but if the set is infinite then it is not the case.

17. Let f : $A \to B$ and g : $B \to C$ be two functions. Then the composition of f and g, denoted by gof is defined as the function gof: $A \to C$ given by

    gof(x): $A \to C$ defined by gof(x) = g(f(x)) $\forall x \in A$



Composition of f and g is written as gof and not fog
gof is defined if the range of f $\subseteq$ domain of f and fog is defined if range of g $\subseteq$ domain of f

18. Composition of functions is not commutative in general

    fog(x) ≠ gof(x).Composition is associative

    If f: X→Y, g: Y→Z and h: Z→S are functions then

    ho(g o f)=(h o g)of

19. A function f: X → Y is defined to be invertible, if there exists a function

    g : Y → X such that gof = I_X and fog = I_Y. The function g is called the inverse of f and is denoted by f^{-1}

20. If f is invertible, then f must be one-one and onto and conversely, if f is one- one and onto, then f must be invertible.

21. If f:A → B and g: B →C are one-one and onto then gof: A → C is also one-one and onto. But If g o f is one –one then only f is one –one g may or may not be one-one. If g o f is onto then g is onto f may or may not be onto.

22. Let f: X → Y and g: Y → Z be two invertible functions. Then gof is also

    Invertible with $(gof)^{-1} = f^{-1} o\, g^{-1}$.

23. If f: R→ R is invertible,

    f(x)=y, then $f^{-1}(y)=x$ and $(f^{-1})^{-1}$ is the function f itself.

24. A binary operation * on a set A is a function from A X A to A.

25. Addition, subtraction and multiplication are binary operations on R, the set of real numbers. Division is not binary on R, however, division is a binary operation on R-{0}, the set of non-zero real numbers

26. A binary operation $*$ on the set X is called commutative, if $a*b = b*a$, for every a,b$\in$X

27. A binary operation $*$ on the set X is called associative, if $a*(b*c) = (a*b)*c$, for every a, b, c$\in$X

28. An element $e \in A$ is called an **identity** of *A* with respect to *, if for each $a \in A$, $a * e = a = e * a$.
    The identity element of (A, *) if it exists, is **unique**.

29. Given a binary operation $*$ from $A \times A \rightarrow A$, with the identity element e in A, an element $a \in A$ is said to be invertible with respect to the operation $*$, if there exists an element b in A such that $a * b = e = b * a$, then b is called the inverse of a and is denoted by a^{-1}.

30. If the operation table is symmetric about the diagonal line then, the operation is commutative.

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

The operation * is commutative.

31. Addition '+' and multiplication '·' on N, the set of natural numbers are binary operations But subtraction '–' and division are not since (4, 5) = 4 - 5 = $-1 \notin N$ and 4/5 = $.8 \notin N$

Chapter: Inverse Trigonometry

Key Concepts

1. Inverse trigonometric functions map real numbers back to angles.

2. Inverse of sine function denoted by $\sin^{-1}$ or arc sin(x) is defined on

   [-1,1] and range could be any of the intervals

   $$\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right], \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

3. The branch of $\sin^{-1}$ function with range $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ is the principal branch.

   So $\sin^{-1}: [-1,1] \rightarrow \left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

4. The graph of $\sin^{-1} x$ is obtained from the graph of sine x by interchanging the x and y axes

5. Graph of the inverse function is the mirror image (i.e reflection) of the original function along the line y = x.

6. Inverse of cosine function denoted by $\cos^{-1}$ or arc cos(x) is defined in [-1,1] and range could be any of the intervals $[-\pi,0]$, $[0,\pi]$,$[\pi,2\pi]$.

   So,$\cos^{-1}: [-1,1] \rightarrow [0,\pi]$.

7. The branch of $\tan^{-1}$ function with range $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ is the principal branch. So $\tan^{-1}: R \rightarrow \left(\frac{-\pi}{2},\frac{\pi}{2}\right)$.

8. The principal branch of $\text{cosec}^{-1} x$ is $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$-{0}.

   $\text{cosec}^{-1} x : R-(-1,1) \rightarrow \left[\frac{-\pi}{2},\frac{\pi}{2}\right]$-{0}.

9. The principal branch of $\sec^{-1} x$ is $[0,\pi]$-{ $\frac{\pi}{2}$ }.

   $\sec^{-1} x : R-(-1,1) \rightarrow [0,\pi]$-{ $\frac{\pi}{2}$ }.

10. $\cot^{-1}$ is defined as a function with domain R and range as any of the intervals $(-\pi,0)$, $(0,\pi)$,$(\pi,2\pi)$. The principal branch is $(0,\pi)$

So $\cot^{-1} : R \to (0,\pi)$

11. The value of an inverse trigonometric function which lies in the range of principal branch is called the principal value of the inverse trigonometric functions.

Key Formulae

1. Domain and range of Various inverse trigonometric Functions

Functions	Domain	Range (Principal Value Branches)
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \text{cosec}^{-1} x$	$R - (-1,1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$y = \sec^{-1} x$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	R	$(0, \pi)$

2. **Self Adjusting property**

$\sin(\sin^{-1}x)=x$; $\sin^{-1}(\sin x) = x$

$\cos(\cos^{-1} x)=x;\cos^{-1}(\cos x)=x$

$\tan(\tan^{-1} x)=x;\tan^{-1}(\tan x)=x$

Holds for all other five trigonometric ratios as well.

3. **Reciprocal Relations**

$\sin^{-1}\left(\frac{1}{x}\right)=\text{cosec}^{-1}\ x,\ x \geq 1 \text{ or } x \leq -1$

$\cos^{-1}\left(\frac{1}{x}\right)=\sec^{-1} x, x \geq 1 \text{ or } x \leq -1$

$\tan^{-1}\left(\frac{1}{x}\right)=\cot^{-1} x, x > 0$

4. Even and Odd Functions

(i) $\sin^{-1}(-x) = -\sin^{-1}(x),\ x \in [-1,1]$

(ii) $\tan^{-1}(-x) = -\tan^{-1}(x),\ x \in R$

(iii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x,\ |x| \geq 1$

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1,1]$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x, |x| \geq 1$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R$

5. Complementary Relations

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, x \in [-1,1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in R$

(iii) $\text{cosec}^{-1}x + \sec^{-1} x = \frac{\pi}{2}, |x| \geq 1$

6. Sum and Difference Formuale

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), xy > -1$

(iii) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$

(iv) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$

(v) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$

(vi) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}[xy + \sqrt{1-x^2}\sqrt{1-y^2}]$

(vii) $\cot^{-1}x + \cot^{-1}y = \cot^{-1}\left(\frac{xy-1}{x+y}\right)$

(viii) $\cot^{-1}x - \cot^{-1}y = \cot^{-1}\left(\frac{xy+1}{y-x}\right)$

7. Double Angle Formuale

(i) $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}, |x| \le 1$

(ii) $2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}, x \ge 0$

(iii) $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}, -1 < x < 1$

(iv) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2}), \frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(v) $2\cos^{-1}x = \sin^{-1}(2x\sqrt{1-x^2}), \frac{1}{\sqrt{2}} \le x \le 1$

8. Conversion Properties

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$

$= \tan^{-1}\frac{x}{x\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$

(ii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$

$= \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}}$

(iii) $\tan^{-1}x = \sin^{-1} = \dfrac{x}{\sqrt{1-x^2}}$

$= \cos^{-1}\dfrac{x}{\sqrt{1+x^2}} = \sec^2\sqrt{1+x^2}$

$= \operatorname{cosec}^{-1}\dfrac{\sqrt{1+x^2}}{x}$

Properties are valid only on the values of x for which the inverse functions are defined.