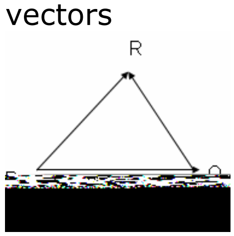
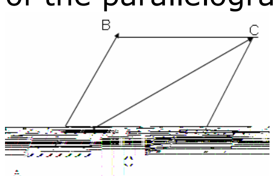
Class XII - Math

Unit: Vectors and Three Dimensional Geometry

Concepts and Formulae

VECTORS

Position Vector	Definition	The position vector of point P ≡ (x_1,y_1,z_1) with respect to the origin is given by: $\overline{OP} = \vec{r} = \sqrt{x^2 + y^2 + z^2}$
Direction cosines	Definition	If the position vector $\overrightarrow{OP}$ of a point P makes angles α, β and γ with x, y and z axis respectively, then α, β and γ are called the **direction angles** and $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **Direction cosines** of the position vector.
Direction ratios	Relation between drs dcs and magnitude of the vector	The magnitude (r), direction ratios (a, b, c) and direction cosines (ℓ, m, n) of any vector are related as: $\ell=\frac{a}{r}$, $m=\frac{m}{r}$, $n=\frac{c}{r}$
Vector Addition	Laws	**Triangle Law**: Suppose two vectors are represented by two sides of a triangle in sequence, then the third closing side of the triangle represents the sum of the two vectors  $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$ **Parallelogram Law**: If two vectors $\vec{a}$ and $\vec{b}$ are represented by two adjacent sides of a parallelogram in magnitude and direction, then their sum $\vec{a}+\vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram.  $\overrightarrow{OA}$??+ $\overrightarrow{OB} = \overrightarrow{OC}$

Prcperties of vector a`dition	Commutative + roperty	For any two vectors $\vec{a}$ and $\vec{b}$, $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
	Associative property	For any three vectors $\vec{a},\vec{b}$ and $\vec{c}$, $\left(\vec{a}+\vec{b}\right)+\vec{c}=\vec{a}+\left(\vec{b}+\vec{c}\right)$
Multiplicatiof of a vec 0or by a scalar	Definition	If $\vec{a}$ is a vector and λ a scalar. Product of vector $\vec{a}$ by the scalar λ is $\lambda\vec{a}$. Also, $\lvert\lambda\vec{a}\rvert = \lvert\lambda\rvert\lvert\vec{a}\rvert$
	Properties	Let $\vec{a}$ and $\vec{b}$ be any two vectors and k and m being two scalars then (i)$k\vec{a}+m\vec{a}=(k+m)\vec{a}$ (ii)$k(m\vec{a})=(km)\vec{a}$ (iii)$k(\vec{a}+\vec{b})=k\vec{a}+k\vec{b}$
Vector joining two points	Definition	The vector $\overrightarrow{P_1P_2}$ joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2$ (O is the origin) is given by: $\overrightarrow{P_1P_2}=\overrightarrow{OP_2}-\overrightarrow{OP_1}$
	Magnitude	The magnitude of vector $\overrightarrow{P_1P_2}$ is given by $\overrightarrow{P_1P_2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$
Component Form		Vector in component form $\vec{r}=x\,\hat{i}+y\,\hat{j}+z\,\hat{k}$ **Equality of vectors** $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ $\vec{a}=\vec{b}$ $a_1=b_1$, $a_2=b_2$ and $a_3=b_3$
	Operations	$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

		Addition of vectors $\vec{a}+\vec{b}=(a_1+\square)\hat{i}+(a_2+b_2)\hat{j}+(a_3+b_3)\hat{k}$ Subtraction of vectors $\vec{a}-\vec{b}=(a_1-b_1)\hat{i}+(a_2-b_2)\hat{j}+(a_3-b_3)\hat{k}$ $\vec{a}$ and $\vec{b}$ are collinear $\vec{b}=\lambda\vec{a}$. where λ is a non zero scalar.
Product of Two Vectors	Scalar (or dot) product of two vectors	Scalar product of two nonzero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}.\vec{b}=\lvert\vec{a}\rvert\lvert\vec{b}\rvert\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le$
	Properties of scalar Product	(i) $\vec{a}\cdot\vec{b}$ is a real number. (ii)If $\vec{a}$ and $\vec{b}$ are non zero vectors then $\vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$. (iii) Scalar product is commutative : $\vec{a}\cdot\vec{b}=\vec{b}.\vec{a}$ (iv)If $\theta=0$ then $\vec{a}\cdot\vec{b}=\lvert\vec{a}\rvert.\lvert\vec{b}\rvert$ (v) If $\theta=\pi$ then $\vec{a}\cdot\vec{b}=-\lvert\vec{a}\rvert.\lvert\vec{b}\rvert$ (vi) scalar product distribute over addition Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors, then $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ (vii)Let $\vec{a}$ and $\vec{b}$ be two vectors, and λ be any scalar. Then $(\lambda\vec{a}).\vec{b}=(\lambda\vec{a}).\vec{b}=\lambda(\vec{a}.\vec{b})=a.(\lambda\vec{b})$ (viii) Angle between two non zero vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta=\dfrac{\vec{a}.\vec{b}}{\lvert\vec{a}\rvert.\lvert\vec{b}\rvert}$
	Projection of a vector	Projection of a vector $\vec{a}$ on other vector $\vec{b}$ is given by $\vec{a}.\hat{b}$ or $\vec{a}.\left(\dfrac{\vec{b}}{\lvert\vec{b}\rvert}\right)$ or $\dfrac{1}{\lvert\vec{b}\rvert}(\vec{a}.\vec{b})$

	Section formula	The position vector of a point R dividing a line segment joi P and Q whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, in (i) internally, is given by $\dfrac{n\vec{a}+m\vec{b}}{m+n}$ (ii) externally, is given by $\dfrac{m\vec{b}-n\vec{a}}{m-n}$
	Inequalities	Cauchy-Schwartz Inequality $\left\|\vec{a}.\vec{b}\right\| \le \left\|\vec{a}\right\|.\left\|\vec{b}\right\|$ Triangle Inequality:
	Vector (or cross) product of two vectors	The vector product of two nonzero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}\times\vec{b}$ and defined as $\vec{a}\times\vec{b}=\|a\|\|b\|\sin\theta\hat{n}$ where, θ is the angle between $\vec{a}$ and $\vec{b}, 0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a},\vec{b}$ and $\hat{n}$ form a right handed system.
	Properties of cross product of vectors	(i) $\vec{a}\times\vec{b}$ is a vector (ii) If $\vec{a}$ and $\vec{b}$ are non zero vectors then $\vec{a}\times\vec{b}=0$ iff $\vec{a}$ and $\vec{b}$ are collinear. (iii) If $\theta=\dfrac{\pi}{2}$, then $\left\|\vec{a}\times\vec{b}\right\|=\left\|\vec{a}\right\|.\left\|\vec{b}\right\|$ (iv) vector product distribute over addition If $\vec{a},\vec{b}$ and $\vec{c}$ are three vectors and λ is a scalar, then (i) $\vec{a}\times(\vec{b}+\vec{c})=\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)$ (ii) $\lambda(\vec{a}\times\vec{b})=(\lambda\vec{a})\times\vec{b}=\vec{a}\times(\lambda\vec{b})$ (v) If we have two vectors $\vec{a}$ and $\vec{b}$ given in component form as $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

THREE DIMENSIONAL GEOMETRY

Direction Cosines	Definition	The direction cosines of the line joining P(x_1,y_1,z_1) and Q(x_2,y_2,z_2) are $\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$ where $PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$
Skew Lines	Definition	**Skew lines** are lines in space which are neither parallel nor intersecting. They lie in different planes.
	Angle between skew lines	**Angle between skew lines** is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.
	Angle between two lines	The angle θ between two vectors $\overline{OA} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\overline{OB} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ is given b $\cos\theta = \left\| \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \right\|$
Equation of a line	Vector Equation	Vector equation of a line that passes through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$
	Cartesian Equation	Direction ratios of the line L are a, b, c. Then, cartesian form of equation of the line L is : $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$
	Equation of line passing through two given points	1) **Vector Equation** The vector equation of a line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$ 2) **Cartesian Equation**

		Cartesian equation of a line that passes through tw points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$
	Condition for perpendicularity	Two lines with direction ratios a_1, a_2, a_3 and b_1, b_2, b_3 respectively are perpendicular if: $a_1b_1 \quad a_2b_2 \quad c_1c_2 \quad 0$
	Condition for parallel lines	Two lines with direction ratios a_1, a_2, a_3 and b_1, b_2, b_3 respectively are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
Shortest Distance between two lines in space	Distance between two skew lines:	1) <u>Vector form:</u> Shortest distance between two skew lines L and m, $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is $d = \left\| \frac{\vec{b_1} \times \vec{b_2} . (\vec{a_2} - \vec{a_1})}{\|\vec{b_1} \times \vec{b_2}\|} \right\|$ 2) <u>Cartesian form</u> The equations of the lines in Cartesian form $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ Then the shortest distance between them is $d = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2 + (c_1a_2-c_2a_1)^2 + (a_1b_2-a_2b_1)^2}}$
	Distance between parallel lines	Distance between parallel lines $\vec{r} = \vec{a_1} + \lambda\vec{b}$ and $\vec{r} = \vec{a_2} + \mu\vec{b}$ is $d = \left\| \frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{\|\vec{b}\|} \right\|$
Equation of plane		In the vector form, equation of a plane which is at a distance d from the origin, and $\hat{n}$ is the unit vector normal to the plane through the origin is

		$\vec{r}.\hat{n} = d$
Equation of plane		Equation of a plane which is at a distance of d from the origin and the direction cosines of the normal to the plane as l, m, n is $lx + my + nz = d$.
Equation of plane		Equation of a plane perpendicular to a given line with direction ratios A, B, C and passing through a given point (x_1, y_1, z_1) is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$
Equation of plane		Equation of a plane passing through three non collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$
Equation of plane	Intercept form of equation of plane.	Equation of a plane that makes intercepts a, b and c with x, y and z-axes respectively is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
Equation of plane	Equation of a plane passing through the intersectio n of two given planes.	Any plane passing thru the intersection of two planes $\vec{r}.\vec{n_1} = d_1$ and $\vec{r}.\vec{n_2} = d_2$ is given by, $\vec{r}.\left(\vec{n_1} + \lambda\vec{n_2}\right) = d_1 + \lambda d_2$
	Coplanarity of two lines	1) <u>Vector form:</u> The given lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ are coplanar if and only $\left(\vec{a_2} - \vec{a_1}\right).\left(\vec{b_1} \times \vec{b_2}\right) = 0$ 2) <u>Cartesian Form</u> Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinates of the points M and N respectively. Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios

		of $\vec{b_1}$ and respectively. The given lines are coplanar if and only if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$
Angle between two planes	Vector form	If $\vec{n_1}$ and $\vec{n_2}$ are normals to the planes $\vec{r}.\vec{n_1}=d_1$ and $\vec{r}.\vec{n_2}=d_2$ and θ is the angle between the normals drawn from some common point. $\cos\theta=\left\|\dfrac{\vec{n_1}.\vec{n_2}}{\|\vec{n_1}\|\|\vec{n_2}\|}\right\|$
	Cartesian form	Let θ is the angle between two planes $A_1x+B_1y+C_1z+D_1=0$, $A_2x+B_2y+C_2z+D_2=0$ The direction ratios of the normal to the planes are □$_1$□□□□□□□$_1$□□□□□□□$_2$□□□$_2$□□□□$_2$. $\cos\theta =$ $\overrightarrow{OP}=\vec{r}=\sqrt{x^2+y^2+z^2}$
Angle between a line and a plane		Let the angle between the line and the normal to the plane = θ $\cos\theta=\left\|\dfrac{\|\vec{b}.\vec{n}\|}{\|\vec{b}\|\|\vec{n}\|}\right\|$
Distance of a point from a plane		Distance of point P with position vector $\vec{a}$ from a plane $\vec{r}.\vec{N}=d$ is $\dfrac{\|\vec{a}.\vec{N}-d\|}{\|\vec{N}\|}$ where $\vec{N}$ is the normal to the plane